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CHARACTERIZATION OF THE COLLAPSING MEROMORPHIC PRODUCTS

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Abstract

Let K be an algebraically closed complete ultrametric field. Let $a \in K$, $r > 0$. We consider a meromorphic product $F(x) = \prod_{n\in\mathbb{N}} \frac{x-a_n}{x-b_n}$, where $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are sequences satisfying $|b_n - a| < r$ whenever $n \in \mathbb{N}$, $\lim\limits_{n\to+\infty} |b_n - a| = r$, $\lim\limits_{n\to\infty} a_n - b_n = 0$ and $\min\limits_{m\neq n} |b_m - b_n| > 0$. We prove that if K has characteristic zero, then F is collapsing if and only if $\sum\limits_{n=0}^{\infty} (a_n)^j - (b_n)^j = 0$ for every $j \in \mathbb{N}$. Moreover, if K has characteristic $\neq 0$, then there exists a meromorphic product f of the form $\prod\limits_{n\in\mathbb{N}} \frac{x-c_n}{x-e_n}$ such that $F(x) = (f(x))^p$ whenever $x \in \{x \in K |\ |x-a| \geq r\}$ if and only if $\sum\limits_{n=0}^{\infty} (a_n)^j - (b_n)^j = 0$ for every $j \in \mathbb{N}$.

Notations and definitions

Let K be an algebraically closed field, complete with respect to an ultrametric absolute value. Given a set D in K, $H(D)$ denotes the set of the analytic elements in D, i.e., the completion of the algebra $R(D)$ of rational functions with no pole in D, with respect to the topology of uniform convergence.

Given $a \in K$ and $r > 0$, $d(a, r)$ (resp. $d(a, r^-)$) denotes the disk $\{x \in K |\ |x - a| \leq r\}$ (resp. $\{x \in K |\ |x - a| < r\}$).

We put $V = d(a, r^-)$ and $E = K \setminus V$. A sequence $(e_n)_{n\in\mathbb{N}}$ in V satisfying $\lim\limits_{n\to\infty} |e_n - a| = r$ and $\min\limits_{m\neq n} |e_m - e_n| > 0$ will be called a *polar sequence associated to* V.

Henceforth, $(b_n)_{n\in\mathbb{N}}$ will denote a polar sequence associated to V and $(a_n)_{n\in\mathbb{N}}$ will denote a sequence in K such that $\lim\limits_{n\to\infty} a_n - b_n = 0$.

For every $x \in K \setminus \{b_0, \dots, b_n, \dots\}$ the product $F_m = \prod\limits_{n=0}^{m} \dfrac{x-a_n}{x-b_n}$ converges to a limit $F(x) = \prod\limits_{n\in\mathbb{N}} \dfrac{x-a_n}{x-b_n}$. Such a function $F(x)$ defined in $K \setminus \{b_1, \dots, b_n, \dots\}$ is called *a meromorphic product associated to the sequence* $(b_n)_{n\in\mathbb{N}}$.

The meromorphic product $\prod\limits_{n\in\mathbb{N}} \dfrac{x-a_n}{x-b_n}$ associated to the sequence $(b_n)_{n\in\mathbb{N}}$ will be said to be *collapsing* if there exists $\ell \in K$ such that F satisfies $\lim\limits_{|x-a|\to r} F(x) = \ell$.

By [**5**], [**7**] it is well known that a meromorphic product f is collapsing if and only if $f-1$ is vanishing along the increasing filter $\mathcal{F}$ of center 0 and diameter 1, and in particular this requires $\mathcal{F}$ to be a T-filter [**4**]. Now, the question whether a meromorphic product is collapsing, in connection with the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, is a quite hard question. Here we will give an answer. In particular, this will be used in the study of the homomorphisms from the group of meromorphic products into the circle $C(0,1)$.

By [**5**], [**7**] we have Lemma a.

Lemma a. *The following are equivalent.*

(1) *F is collapsing,*
(2) $\lim\limits_{|x-a|\to r} F(x) = 1$,
(3) *$F(x) = 1$ whenever $x \in E$.*

Next result is taken from [**5**].

Theorem 0. *Let $f \in H(E)$ satisfy $\lim\limits_{|x|\to\infty} f(x) = 1$ and $\|f-1\|_E < 1$. Let $\epsilon \in]0, \|f-1\|_E[$. There exist a polar sequence $(e_n)_{n\in\mathbb{N}}$ associated to V, together with a meromorphic product $\prod\limits_{n=0}^{\infty} \dfrac{x-c_n}{x-e_n}$ associated to the sequence $(e_n)_{n\in\mathbb{N}}$, satisfying further $|c_n - e_n| < r(\|f-1\|_E + \epsilon)$, and $\prod\limits_{n=0}^{\infty} \dfrac{x-c_n}{x-e_n} = f(x)$ whenever $x \in E$.*

We notice that for every $j \in \mathbb{N}^*$ the series $\sum_{n=0}^{\infty} a_n^j - b_n^j$ is convergent. Lemma b below is easy and will be used in proving Lemma c.

Lemma b. *Let $\lambda \in K$. The following are equivalent.*

i) $\sum_{n=0}^{\infty} a_n^j - b_n^j = 0$ *for every* $j \in \mathbb{N}^*$

ii) $\sum_{n=0}^{\infty} (a_n + \lambda)^j - (b_n + \lambda)^j = 0$ *for every* $j \in \mathbb{N}^*$.

Lemma c. *F satisfies $F'(x) = 0$ for all $x \in E$ if and only if for every $j \in \mathbb{N}^*$ the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ satisfy* $\sum_{n=0}^{\infty} (a_n)^j - (b_n)^j = 0$.

Proof: By Lemma b we may clearly assume $a = 0$ without loss of generality. Let $r' \in [r, +\infty[$ be such that $|a_n| < r'$ for every $n \in \mathbb{N}$, and let $E' = K \setminus d(0, r'^-)$. We can see that $\|F - 1\|'_E \leq \sup_{n\in\mathbb{N}} \frac{|b_n - a_n|}{r'} < 1$. Hence $\frac{F'}{F}$ obviously belongs to $H(E')$. Let $g = \frac{F'}{F}$. It is seen that $g(x) = \sum_{n=0}^{\infty} \frac{1}{x - a_n} - \frac{1}{x - b_n}$. For each α, $\beta \in V$, and for every $x \in E'$ we have

$$\frac{1}{x - \alpha} - \frac{1}{x - \beta} = \sum_{j=0}^{\infty} \frac{\alpha^j - \beta^j}{x^{j+1}} = \sum_{j=1}^{\infty} \frac{\alpha^j - \beta^j}{x^{j+1}}.$$

Applying this to each term $\frac{1}{x-a_n} - \frac{1}{x-b_n}$, we obtain

$$g(x) = \sum_{n=0}^{\infty} \left(\sum_{j=1}^{\infty} \frac{(a_n)^j - (b_n)^j}{x^{j+1}} \right)$$

for all $x \in E'$. Now, let us fix $x \in E'$. We see that when j tends to $+\infty$, the convergence of $\frac{(a_n)^j - (b_n)^j}{x^{j+1}}$ to 0 is uniform with respect to n. Hence we have

$$g(x) = \sum_{j=1}^{\infty} \left[\sum_{n=0}^{\infty} \frac{(a_n)^j - (b_n)^j}{x^{j+1}} \right].$$

But now, this holds for any $x \in E'$. Besides, as F belongs to $H(E)$, we know that its Mittag-Leffler series [**3**], [**4**] is the same in $H(E)$ and in $H(E')$, hence this is the Mittag-Leffler series of F in $H(E)$. Hence we see that $F'(x) = 0$ if and only if the Mittag-Leffler series of g is identically equal to 0, i.e.: $\sum_{n=0}^{\infty} (a_n)^j - (b_n)^j = 0$ for every $j \in \mathbb{N}^*$. This ends the proof. ■

Now, we can conclude

Theorem 1. *K is supposed to have characteristic zero. Then F is collapsing if and only if for every $j \in \mathbb{N}^*$ the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ satisfy $(\mathcal{E}_j) \sum_{n=0}^{\infty} (a_n)^j - (b_n)^j = 0$.*

Proof: Indeed, since K has characteristic zero, by [**1**] we know that $F'(x)$ is identically zero in E if and only if $F(x)$ is a constant in E, i.e., F is collapsing. ■

Theorem 2. *Assume K to be of characteristic $p \neq 0$. There exists a polar sequence $(e_n)_{n\in\mathbb{N}}$ associated to V, and a meromorphic product $f(x) = \prod_{n\in\mathbb{N}} \frac{x - c_n}{x - e_n}$, associated to the sequence $(e_n)_{n\in\mathbb{N}}$, satisfying $F(x) = (f(x))^p$ whenever $x \in E$ if and only if for every $j \in \mathbb{N}^*$ the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ satisfy $(\mathcal{E}_j) \sum_{n=0}^{\infty} (a_n)^j - (b_n)^j = 0$.*

Proof: If there exists a meromorphic product f associated to the sequence $(b_n)_{n\in\mathbb{N}}$ such that $(f(x))^p = F(x)$ for all $x \in E$, then obviously we have $F'(x) = 0$ for all $x \in E$, and therefore, by Lemma c, we have $(\mathcal{E}_j) \sum_{n=0}^{\infty} (a_n)^j - (b_n)^j = 0$ for every $j \in \mathbb{N}^*$.

Reciprocally, we suppose Relations $(\mathcal{E}_j)$ satisfied. By Lemma b we have $F'(x) = 0$ for all $x \in E$. Hence, there exists $g \in H(E)$ such that $(g(x))^p = F(x)$ for all $x \in E$. Besides, since F is a meromorphic product associated to the sequence $(b_n)_{n\in\mathbb{N}}$, we notice that $\lim_{|x|\to+\infty} F(x) = 1$. As a consequence, we can choose g such that $\lim_{|x|\to+\infty} g(x) = 1$. Further, it is seen that $g^p = \big((g-1)+1\big)^p = (g-1)^p + 1$, and therefore we have

$$(1) \qquad \|F-1\|_E = \big(\|g-1\|_E\big)^p,$$

hence $\|g-1\|_E < 1$. Let $\epsilon \in]0,1[$. Then by (1) and by Theorem 0 there does exists a polar sequence $(e_n)_{n\in\mathbb{N}}$ associated to V, and a meromorphic product f of the form $\prod_{n\in\mathbb{N}} \frac{x-c_n}{x-e_n}$ such that $f(x) = g(x)$ whenever $x \in E$, and such that $|e_n - c_n| \le \sqrt[p]{\|F-1\|_E} + \epsilon$. This ends the proof. ■

Remark. In [**5**], and [**6**] it was shown how one can construct a collapsing meromorphic product, with the help of certain unbounded functions analytic in the disk $d(0, r^-)$.

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References

1. A. ESCASSUT, Derivative of analytic elements on infraconnected clopen sets, *Indag. Math.* **51** (1989), 63–70.
2. A. ESCASSUT, T-filtres, ensembles analytiques et transformation de Fourier p-adique, *Ann. Inst. Fourier* **25(2)** (1975), 45–80.
3. A. ESCASSUT AND M.-C. SARMANT, Mittag-Leffler series and Motzkin products for invertible analytic elements, *Rivista di Matematica Pura ed Applicata* **12** (1992), 61–73.
4. M. KRASNER, Prolongement analytique uniforme et multiforme dans les corps valués complets. Les tendances géométriques en algèbre et théorie des nombres, Clermont-Ferrand (1964), 94–141; Centre National de la Recherche Scientifique (Colloques internationaux de C.N.R.S. Paris, 143) (1966).
5. M.-C. SARMANT, Produits Méromorphes, *Bull. Sci. Math.* **109** (1985), 155–178.
6. M.-C. SARMANT, Fonctions analytiques et produits croulants, *Collectanea Mathematica* **XXXVI(2)** (1985), 199–218.
7. M.-C. SARMANT, Produits meromorphes et prolongement analytique, Thèse de Doctorat d'Etat, Université Pierre et Marie Curie (1987).

8. M.-C. Sarmant, Factorisation en produit méromorphe d'un elémént semi-inversible, *Bull. Sci. Math.* **115** (1991), 379–394.

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